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Nash Equilibrium Allocations in a Multiple Public Goods Economy

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Abstract

We study allocations of multiple public goods attainable through Nash equilibrium of voluntary contribution of non-consumable private resourses. We characterize Nash equilibrium allocation and establish a version of the fundamental theorem of welfare economics in an economy with multiple public goods.

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1 Introduction

Resource allocation in a public goods economy is central in public economics. When there are multiple public goods, the problem is quite complex. We have to examine how to allocate scarce resources not only between private consumption and public goods, but also among the public goods.

In this paper, we consider a resource allocation problem in an economy with multiple public goods. The allocation mechanism is simple voluntary contribution (private provision of resources). As a specific feature of our model, we assume private resources are non-consumable. This is because we focus on resource allocation among the public goods. If there is only one public good, then the problem is trivial: All agents contribute all their resources to public good, so an optimal allocation is always achieved. But if there are multiple public goods, the problem becomes nontrivial even if private resources are non-consumable. The basic problem, such as "whether or not is Nash equilibrium allocation Pareto optimal?", is unsolved in this framework.

This paper investigate the following classical problems of welfare economics in a public goods economy with non-consumable resources.

(A) Are Nash equilibrium allocations Pareto optimal? If the answer is "no" in general, when is it "yes"?

(B) Are any Pareto optimal allocations attainable through Nash equilibrium when redistribution of private resources is possible?

We provide answers to the problems by establishing a version of fundamental welfare theorems. With respect to (A), we derive several sufficient conditions for Nash equilibrium allocations to be Pareto optimal (Proposition 1,2, and 3). To deal with (B), we first give a characterization of allocations attainable through Nash equilibrium with transfaer (Theorem 2). This result enables us to derive a necessary and sufficient condition for any Pareto optimal allocation to be achieved by Nash equilibrium with transfer (Proposition 4).

Our model may be regarded as income redistribution game formulated by Nakayama (1980) if the number of public goods is equal to the one of agents in the economy [1]. Nakayama (1980) provide sufficient conditions for Nash equilibrium of income redistribution games to be Pareto optimal. We extend his results to the public goods economy. Our results admit the possibity of productions, and are independent of the number of the agents in the economy.

The paper is organized as follows. In section 2, the basic model is introduced, and the definition of Pareto optimality in our model is provided. Section 3 contains the formal description of our games. Section 4 introduces allocations corresponding to Nash equilibrium of the game, and gives a necessary condition for Nash equilibrium allocation. Section 5 contains several sufficient conditions for the first welfare theorem. In section 6, we investigate the possibility of the second welfare theorem in our economy. Section 7 extends our model to the case of consumable private resources, and discuss the results.

2 The Aumann-Kurz-Neyman Economy

Our description of the economy with public goods follows Aumann, Kurz, and Neyman [1], [2]. There are one non-consumable resources and m public goods. Public goods are produced from resources. The production technology is represented by a production function $F : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ such that for any $x = (x_1, \ldots, x_m) \in \mathbb{R}^m_+$

$$F(x) = \begin{pmatrix} f_1(x_1) \\ \vdots \\ f_m(x_m) \end{pmatrix},$$

where x_i is resource input and f_i is production function for public good i ($i = 1, \ldots, m$). Let H be a set of agents whose cardinality is n, i.e., $|H| = n$. The agent h is characterized by the pair (u_h, e_h) of utility function $u_h : \mathbb{R}^m_+ \to \mathbb{R}$ and initial endowment of resources e_h.

Definition 1. A *public goods economy* $\mathcal{E}$ is a list of the set of agents H, the

[1] Then the public good i should be interpreted as the redistributed income of agent i.

production technology F, and the agents' characteristics (u_h, e_h):

$$\mathcal{E} = \Big(H, F, (u_h, e_h)_{h \in H}\Big).$$

We shall assume the standard convex environments.

Assumption 1.

(i) f_i is increasing, continuous, and concave for $i = 1, \ldots, m$.

(ii) u_h is increasing, continuous and strictly quasi concave for every $h \in H$.

(iii) $e_h \geq 0$ for all $h \in H$.

Let $\bar{e}$ be the total resources of $\mathcal{E}$; $\bar{e} := \sum_h e_h$. The set of feasible resource input vectors is denoted by $C(\bar{e})$:

$$C(\bar{e}) := \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m_+ \;\middle|\; \sum_{i=1}^m x_i \leq \bar{e} \right\}$$

The feasible set of public goods bundle in $\mathcal{E}$ is denoted by $\mathcal{A}(\mathcal{E})$:

$$\mathcal{A}(\mathcal{E}) := \{ g \in \mathbb{R}^m_+ \mid g \leq F(x) \text{ for some } x \in C(\bar{e}) \}.$$

Definition 2. A feasible allocation g is *Pareto optimal* if there exists no $g' \in \mathcal{A}(\mathcal{E})$ such that $u_h(g') \geq u_h(g)$ for any $h \in H$ and $u_h(g') > u_h(g)$ for some $h \in H$.

3 The Public Goods Games

In this section, we formulate voluntary contribution to public goods as a strategic form game. Let x_i^h be a contribution (resource input) to public good i of agent h, and $x^h = (x_1^h, \ldots, x_m^h)$ be a contribution vector of agent h.

Given total resources $\bar{e}$, the set of feasible resource allocation vectors is denoted by $T(\bar{e})$;

$$T(\bar{e}) := \left\{ t = (t_h)_{h \in H} \in \mathbb{R}^n_+ \;\middle|\; \sum_{h \in H} t_h = \bar{e} \right\}.$$

Given the resource allocation $t \in T(\bar{e})$, a contribution x^h of agent h is feasible if $\sum_i x_i^h \leq t_h$. Let $X_h(t_h)$ be a set of feasible contribution of agent h;

$$X_h(t_h) := \left\{ x^h = (x_1^h, \ldots, x_m^h) \in \mathbb{R}^m_+ \;\middle|\; \sum_{i=1}^{m} x_i^h \leq t_h \right\}.$$

Let $X(t)$ be a set of profiles of feasible contributions; $X(t) := \prod_{h \in H} X_h(t_h)$. Note that $X_h(t_h)$ is compact and convex, so is $X(t)$.

For any $x \in X(t)$, let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$ be a total contribution (aggregate resource inputs); $\bar{x} := \sum_{h \in H} x^h$.

The payoffs are defined as follows:

$$U_h(x) := u_h \circ F(\bar{x}) = u_h\left(f_1\left(\sum_{j \in H} x_1^j \right), \ldots, f_m\left(\sum_{j \in H} x_m^j \right) \right) \quad \text{for } h \in H$$

$$U(x) := \prod_{h \in H} U_h(x).$$

For any $x = (x^h)_{h \in H} \in X(t)$, let x^{-h} be the list $(x^k)_{k \in H \setminus \{h\}}$.

Definition 3. Given $t \in T(\bar{e})$, a *public goods game* $\mathcal{G}_t(\mathcal{E})$ consists of the set of player H, the set of strategy profiles $X(t)$, and the payoff function U:

$$\mathcal{G}_t(\mathcal{E}) = (H, X(t), U).$$

Definition 4. A strategy profile $x \in X(t)$ is *Nash equilibrium* of $\mathcal{G}_t(\mathcal{E})$ if for any $h \in H$ and any $y^h \in X_h(t_h)$

$$U_h(x) \geq U_h(y^h, x^{-h})$$

For consistency, we shall prove the existence of Nash equilibrium of our games via a standard fixed point argument.

Theorem 1. *There exists a Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$ for any $t \in T(\bar{e})$.*

Proof . Note that $U_h(x)$ is continuous and strictly quasi concave. Followig standard arguments, we define the best reply functions such that for any $x \in X(t)$,

$$b_h(x^{-h}) := \arg\max\{\ U_h(y^h, x^{-h}) \mid y^h \in X_h(t_h)\ \}.$$
$$B(x) := \prod_{h \in H} b_h(x^{-h}).$$

$B : X(t) \to X(t)$ is continuous and $X(t)$ is compact and convex. By Brouwer's fixed point theorem, there exists $x \in X(t)$ such that $x = B(x)$. Then x is Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$. □

Remark 1. In general, a Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$ is *not* unique for $t \in T(\bar{e})$.

4 Nash Equilibrium Allocation

The goal of this section is to provide the necessary condition for an allocation to be attainable through Nash equilibrium given a resource distribution (Lemma 1). We first introuduce the concept of *Nash equilibrium allocation* which is the one corresponding to Nash equilibrium of the public goods game.

Definition 5. Given $t \in T(\bar{e})$, a public goods bundle $g = (g_1, \dots, g_m)$ is *Nash equilibrium allocation* of $\mathcal{G}_t(\mathcal{E})$ if $g = F(\bar{x})$ for some Nash equilbrium x of $\mathcal{G}_t(\mathcal{E})$ where $\bar{x} = \sum_{h \in H} x^h$.

Given $g \in \mathcal{A}(\mathcal{E})$, we define constrained feasible set $\mathcal{A}_g(\mathcal{E})$ as follows:

$$\mathcal{A}_g(\mathcal{E}) := \{g' \in \mathcal{A}(\mathcal{E}) \mid g'_i \geq g_i \text{ for all } i\ \}.$$

Note that $\mathcal{A}(\mathcal{E})$ is compact in $\mathbb{R}^m$ and so is $\mathcal{A}_g(\mathcal{E})$ for any $g \in \mathcal{A}(\mathcal{E})$. Given $g \in \mathbb{R}^m_+$ and $J \subset \{1, \dots, m\}$, we define $g(J) = (g_1(J), \dots, g_m(J)) \in \mathbb{R}^m_+$ as follows;

$$g_i(J) := \begin{cases} g_i & (i \notin J) \\ 0 & (i \in J). \end{cases}$$

For notational simplicity, we write $g(i)$ instead of $g(\{i\})$.

Let us define the key concep in the following discussion.

Definition 6. Let $g \in \mathcal{A}(\mathcal{E})$ and $I := \{i \mid g_i > 0\}$. g satisfies *condition* **(M)** if for any $i \in I$, there exists $h \in H$ such that $u_h(g) \geq u_h(g')$ for any $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$.

If an allocation g satisfies (M), then g may be regarded as a common solution of several utility maximization problem over some constrained feasible set $\mathcal{A}_{g(i)}(\mathcal{E})$.

The next lemma states that (M) is the necessary condition for Nash equilibrium allocation.

Lemma 1. *Given $t \in T(\bar{e})$, let g be a Nash equilibrium allocation of $\mathcal{G}_t(\mathcal{E})$. Then g satisfies condition (M).*

Proof . If g is a Nash equilibrium allocation, then $u_h(g) \geq u_h(g')$ for any $g' \in \mathcal{A}_{g^h}(\mathcal{E})$ and for all $h \in H$, where $g^h := F(\bar{x} - x^h)$ and $x \in X(t)$ is Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$. Let us define $I_h := \{i \mid g_i > g_i^h\}$.

We will show that g is a maximizer of u_h over $\mathcal{A}_{g(I_h)}(\mathcal{E})$. Suppose $u_h(g') > u_h(g)$ for some $g' \in \mathcal{A}_{g(I_h)}(\mathcal{E})$. Then $g(\varepsilon) := \varepsilon \cdot g' + (1 - \varepsilon) \cdot g \in \mathcal{A}_{g^h}(\mathcal{E})$ for suffiently small ε and $u_h(g(\varepsilon)) > u_h(g)$ by quasi concavity of u_h, which is contradiction.

Since $g_i = 0$ for $i \notin \bigcup_h I_h$, $I = \bigcup_h I_h$. Therefore for any $i \in I$, there exists $h \in H$ such that $i \in I_h$, which implies the result. $\square$

5 Optimality of Nash equilibrium Allocations

In this section, we provide several sufficient conditions for Nash equilibrium allocation to be Pareto optimal.

Proposition 1. *If all the agents' preferences for public goods bundle are identical, then there is a unique Nash equilibrium allocation of $\mathcal{G}_t(\mathcal{E})$ for any $t \in T(\bar{e})$*

Proof . Suppose all the agents' preferences are identical. Then there exists the (representative) utility function $u(g)$ such that $u(g) = u_h(g)$ for all $h \in H$. Let g^*

be Nash equilibrium allocation and $I := \{i \mid g_i^* > 0\}$. By Lemma 1, g^* satisfies condition (M). Under the identical preference assumption, (M) is equivalent to that $u(g^*) \geq u(g)$ for any $g \in \mathcal{A}_{g^*(i)}(\mathcal{E})$ and any $i \in I$. We will show that $u(g^*) \geq u(g)$ for any $g \in \mathcal{A}(\mathcal{E})$. Suppose the contrary. Then there exists $g' \in \mathcal{A}(\mathcal{E})$ such that $u(g') > u(g^*)$. It follows from (M) that $g' \notin \mathcal{A}_{g^*(i)}(\mathcal{E})$ for any $i \in I$, which implies $g'_j < g^*_j$ for some $j \notin I$. It follows from the fact that $g^*_j = 0$ for $j \notin I$ that $g' \notin \mathcal{A}(\mathcal{E})$, which is contradiction. Therefore $g^* \in \arg\max\{u(g) \mid g \in \mathcal{A}(\mathcal{E})\ \}$. The maximizer g^* must be unique by strict quasi concavity of $u(g)$. □

The next two results are extensions of the sufficient conditions for Pareto optimality in Nakayama (1980) to the public goods economy. Note that our results are independent of the number of agents.

Proposition 2. *Suppose x is Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$ and g is the corresponding Nash equilibrium allocation. If there exists $h \in H$ such that $x_i^h > 0$ for all $i \in I = \{i \mid g_i > 0\ \}$, then g is Pareto optimal.*

Proof . Let a be the agent such that $x_i^a > 0$ for all $i \in I$. By the proof of Lemma 1, $u_a(g) \geq u_a(g')$ for any $g' \in \mathcal{A}_{g(I)}(\mathcal{E})$. It follows from $\mathcal{A}_{g(I)}(\mathcal{E}) = \mathcal{A}(\mathcal{E})$ that there is no $g' \in \mathcal{A}(\mathcal{E})$ that Pareto dominates g. □

Proposition 3. *In the case of $m = 2$, any Nash equilibrium allocation of $\mathcal{G}_t(\mathcal{E})$ is Pareto optimal for any $t \in T(\bar{e})$.*

Proof . Let g be Nash equilibrium allocation of $\mathcal{G}_t(\mathcal{E})$ which is not Pareto optimal. Then there exists g' such that $u_h(g') > u_h(g)$ for all $h \in H$. Suppose $g \in \mathbb{R}^2_{++}$. Without loss of generality, we assume $g' \in \mathcal{A}_{g(2)}(\mathcal{E})$. This implies $g \notin \arg\max u_h(g'')$ subject to $g'' \in \mathcal{A}_{g(2)}(\mathcal{E})$ for any $h \in H$, which contradicts Lemma 1. If $g_1 = 0$ and $g_2 > 0$, then $g' \in \mathcal{A}_{g(2)}(\mathcal{E})$. So $g \notin \arg\max u_h(g'')$ subject to $g'' \in \mathcal{A}_{g(2)}(\mathcal{E})$ for any $h \in H$, which contradicts Lemma 1. If $g_1 > 0$ and $g_2 = 0$, then $g' \in \mathcal{A}_{g(1)}(\mathcal{E})$. So $g \notin \arg\max u_h(g'')$ subject to $g'' \in \mathcal{A}_{g(1)}(\mathcal{E})$ for any $h \in H$, which contradicts Lemma 1. □

6 Decentralization of Pareto Optimal Allocation

In this section, we consider implementability of Pareto optimal allocations by Nash equilibrium through resource redistribution (transfer). At first, we define the concept of Nash equilibrium allocation under redistribution of private resources.

Definition 7. A public goods bundle g is *Nash equilibrium allocation with transfer* in $\mathcal{E}$ if there exists $t \in T(\bar{e})$ such that g is Nash equilibrium allocation of $\mathcal{G}_t(\mathcal{E})$.

In general, the second welfare theorem does not hold in our economy: some Pareto optimal allocation may not be Nash equilibrium allocation with transfer. In the following theorem, we provide characterization of the allocations attainable through Nash equilibrium with resource redistribution.

Theorem 2. *Let g be a feasible allocation of public goods in $\mathcal{E}$. Then g is Nash equilibrium allocation with transfer if and only if g satisfies condition (M).*

Proof . The "only if" part of the theorem is direct consequence of Lemma 1. We will show the converse. Suppose g is a feasible allocation in $\mathcal{E}$. Then there exists resource inputs $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \in C(\bar{e})$ such that $g = F(\bar{x})$. Suppose that for every i there exists $h_i \in H$ such that $u_{h_i}(g) \geq u_{h_i}(g)$ for any $g \in \mathcal{A}_{g(i)}(\mathcal{E})$. We define the resource allocation vector $t = (t^h)_{h \in H}$ in the following way;

$$t_h = \sum_{h=h_i} \bar{x}_i$$

where $t_h = 0$ if $h \neq h_i$ for all i. Given t, consider the strategy profile $x = (x^h)_{h \in H} \in X(t)$ such that

$$x_j^h = \begin{cases} \bar{x}_i & \text{if } j = i \text{ and } h = h_i \\ 0 & \text{otherwise.} \end{cases}$$

We show that the strategy profile x is a Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$. For this, it is sufficient to show that x^h is best reply to x^{-h} for $h \in H$. Fix $h \in H$ arbitrarily. If $h = h_i$ for some i, then the feasible set of public goods by agent h

with his resources t_h is $\mathcal{A}_{g(i)}(\mathcal{E})$, given the contributions x^{-h} of other agents. By the assumption, $u_h(g) \geq u_h(g')$ for $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$. Therefore x^h maximize $U_h(x^h, x^{-h})$; x^h is a best reply to x^{-h}. If $h \neq h_i$ for all i, then $x^h = (0, \ldots, 0)$, $t_h = 0$ and $X_h(t_h) = \{(0, \ldots, 0)\}$. It is clear that x^h is best reply to x^{-h}. So x is a Nash equilibrium of $\mathcal{G}_t(\mathcal{E})$. Therefore g is a Nash equilibrium allocation with transfer in $\mathcal{E}$. □

Note that (M) is the necessary condition for Nash equilibrium allocation by Lemma 1. The theorem above states that (M) is also sufficient condition for Nash equilibrium allocation with transfer.

We consider the following question,"When is any Pareto optimal allocation Nash equilibrium allocation with transfer?" By Theorem 2, the answer is as follows: "If any Pareto optimal allocation satisfies (M), then the second welfare theorem holds". To understand the situation more deeply, we look at (M) from a slightly different angle. We paraphrase (M) in terms of diversity of preferences of agents in the economy.

Definition 8. Fix $g \in \mathcal{A}(\mathcal{E})$ arbitrarily. The economy $\mathcal{E}$ satisfies *Local Non-Diversity condition at g (g-LND)* if the following condition holds: if for some i and for every $h \in H$, there exists $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$ such that $u_h(g') > u_h(g)$, then there exists $g'' \in \mathcal{A}_{g(i)}(\mathcal{E})$ such that $u_h(g'') > u_h(g)$ for all h. The economy $\mathcal{E}$ satisfies *Local Non-Diversity condition (LND)* if $\mathcal{E}$ satisfies g-LND for all Pareto optimal allocation g.

Remark 2. In the above definition, g' may be different from every agent (g' may depend on the index of the agents). But g'' must be independent of the index of agents.

Suppose that for every $h \in H$, there exists $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$ such that $u_h(g') > u_h(g)$ for some i at g.[2] Then all agents may agree to reduce the level of g_i ("the supply of the public good i is too much at g"). If LND is satisfied at g, then all

[2] $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$ is equivalent to $g'_i < g_i$ when $g' \neq g$.

the agents can agree with the common plan g'' how to reduce g_i. In this sense, the preferences among the agents may not be so diverse on the local area $\mathcal{A}_{g(i)}(\mathcal{E})$.

The following theorem states that the second welfare theorem holds in the economy such that the agents' prefereces for public goods are not so diverse at Pareto optimal allocations.

Theorem 3. *Any Pareto optimal allocation is Nash equilibrium allocation with transfer in the economy $\mathcal{E}$ if and only if $\mathcal{E}$ satisfies LND.*

Proof . Suppose $\mathcal{E}$ satisfies LND. Let g be a Pareto optimal allocation. If g is not Nash equilibrium allocation with transfer, then it follows from Theorem 2 that for every $h \in H$ there exists $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$ such that $u_h(g') > u_h(g)$ for some i. By LND, there exists $g'' \in \mathcal{A}(\mathcal{E})$ such that $u_h(g'') > u_h(g)$ for all h. That contradicts Pareto optimality of g.

Suppose $\mathcal{E}$ does not satisfy LND. Then for every $h \in H$, there exists $g' \in \mathcal{A}_{g(i)}(\mathcal{E})$ such that $u_h(g') > u_h(g)$ at some Pareto optimal g and for some i. This implies that g is not maximizer of $u_h(g)$ over $\mathcal{A}_{g(i)}(\mathcal{E})$ for all $h \in H$. It follows from Theorem 2 that g is not Nash equilibrium allocation with transfer. $\square$

Corollary 1. *Suppose $m = 2$. Then any Pareto optimal allocation in $\mathcal{E}$ is Nash equilibrium allocation with transfer in $\mathcal{E}$.*

Proof . In the case of $m = 2$, any economy $\mathcal{E}$ always satisfies LND. $\square$

Corollary 2. *Suppose all the agents' preferences for public goods bundle are identical. Then any Pareto optimal allocation in $\mathcal{E}$ is Nash equilibrium allocation with transfer in $\mathcal{E}$.*

Proof . Identical preference of agents implies that $\mathcal{E}$ satisfies LND. $\square$

7 An Extension

In this section, we extend our model to the general one including consumable private resources. The resource-consumption of agent h is denoted by z_h. Let denote $z = (z_h)_{h\in H}$. The extended utility function of agent h is denoted by $\tilde{u}_h(z_h, g)$. An extended public goods economy $\tilde{\mathcal{E}}$ is a list $(H, (\tilde{u}_h, e_h)_{h\in H})$. An extended public goods game is defined as $(H, X, \tilde{U})$ where $\tilde{U}(x) := \prod_{h\in H} \tilde{U}_h(x)$ and $\tilde{U}_h(x) := \tilde{u}_h(t_h - \sum_{i=1}^m x_i^h, F(\bar{x}))$ where $\sum_{i=1}^m x_i^h \leq t_h$.

An allocation $(z, g) \in \mathbb{R}_+^{n+m}$ is feasible if $\sum_h z_h + x \leq \bar{e}$ and $g \leq F(x)$ for some $x \in C(\bar{e})$. The set of feasible allocations in $\tilde{\mathcal{E}}$ is denoted by $\mathcal{A}(\tilde{\mathcal{E}})$. Let $\bar{x} \in \mathbb{R}_+^m$ be a feasible resource input vector in $\mathcal{E}$, i.e., $\bar{x} \leq \bar{e}$. We define a constrained feasible set with respect to $\bar{x}$, $\mathcal{A}^{\bar{x}}(\mathcal{E})$, such as

$$\mathcal{A}^{\bar{x}}(\tilde{\mathcal{E}}) := \left\{ (z, g) \in \mathbb{R}_+^{n+m} \;\middle|\; \sum_{h\in H} z_h + x \leq \bar{e} \text{ and } g \leq F(x) \text{ for some } x \in C(\bar{x}) \right\}.$$

Definition 9. Let $\bar{x} \in \mathbb{R}_+^m$ be a feasible resource input vector in $\mathcal{E}$. A feasible allocation (z, g) is *constrained Pareto optimal* if there exists no $(z, g') \in \mathcal{A}^{\bar{x}}(\tilde{\mathcal{E}})$ such that $\tilde{u}_h(z_h, g') \geq \tilde{u}_h(z_h, g)$ for all $h \in H$ and $\tilde{u}_h(z_h, g') > \tilde{u}_h(z_h, g)$ for some $h \in H$.

Remark 3. The private consumption allocation $z = (z_h)_{h\in H}$ is fixed in this definition.

Proposition 4. *Suppose $m = 2$. Then any Nash equilibrium allocation of $\mathcal{G}_t(\tilde{\mathcal{E}})$ is constrained Pareto optimal.*

Proof . Let $x^* = (x^{h*})$ be any Nash equilibrium of $\mathcal{G}_t(\tilde{\mathcal{E}})$ and (z^*, g^*) be the corresponding equilibrium allocation. Let us define $\hat{e}_h := \sum_i x_i^{h*}$ and $\hat{e} := (\hat{e}_h)_{h\in H}$. Now consider the auxiliary economy $\hat{\mathcal{E}} = (H, F, (v_h, \hat{e}_h)_{h\in H})$ and the induced game $\mathcal{G}_{\hat{e}}(\hat{\mathcal{E}})$ with payoff function $V_h(x) = \tilde{u}_h(z_h^*, F(\bar{x})$. Then x^* is Nash equilibrium of $\mathcal{G}_{\hat{e}}(\hat{\mathcal{E}})$. For if there exists $h \in H$ such that $V_h(y^h, x^{-h*}) > V_h(x^*)$ for some

$y^h \in X_h(\hat{e}_h)$, then

$$\tilde{U}_h(y^h, x^{-h*}) = \tilde{u}_h(z_h^*, g') = V_h(y^h, x^{-h*}) > V_h(x^*) = \tilde{u}_h(z_h^*, g^*) = \tilde{U}_h(x^*),$$

where $g' := F(y^h + \sum_{k \neq h} x^{k*})$. That contradicts the fact that x^* is Nash equilibrium of $\mathcal{G}_t(\tilde{\mathcal{E}})$. Therefore g^* is Nash equilibrium allocation of $\hat{\mathcal{E}}$. By Proposition 3, g^* is Pareto optimal in $\hat{\mathcal{E}}$, which implies g^* is constrained Pareto optimal in the original economy $\tilde{\mathcal{E}}$. □

Next, we consider a modified version of constrained optimality.

Definition 10. Let $\bar{x} \in \mathbb{R}^m_+$ be a feasible resource input vector in $\mathcal{E}$. A feasible allocation (z, g) is *constrained Pareto optimal with respect to* $\bar{x}$ if there exists no $(z', g') \in \mathcal{A}^{\bar{x}}(\tilde{\mathcal{E}})$ such that $\tilde{u}_h(z'_h, g') \geq \tilde{u}_h(z_h, g)$ for all $h \in H$ and $\tilde{u}_h(z'_h, g') > \tilde{u}_h(z_h, g)$ for all $h \in H$.

Remark 4. In this definition, the aggregate inputs $\bar{x} = (\bar{x}_1, \bar{x}_m)$ is fixed, and redistribution of private consumptions $z = (z_h)_{h \in H}$ is permitted.

If we adopt this modified definition, then a Nash equilibrium allocation is not constrained optimal with respect to the corresponding equilibrium resource inputs in general even if $m = 2$.

For simplicity, we assume the utility functions (preferences) of every agent is quasi linear, i.e., $\tilde{u}_h(z_h, g) = z_h + v_h(g)$ where v_h is concave. Then the feasible allocation (z^*, g^*) is constrained Pareto optimal with respect to $\bar{x}$ if and only if

$$(z^*, g^*) \in \arg\max \left\{ \sum_{h \in H} \tilde{u}_h(z_h, g) \;\middle|\; (z, g) \in \mathcal{A}^{\bar{x}}(\tilde{\mathcal{E}}) \right\}.$$

That is equivalent to

$$g^* \in \arg\max \left\{ \sum_{h \in H} v_h(g) \;\middle|\; \exists z : \; (z, g) \in \mathcal{A}^{\bar{x}}(\tilde{\mathcal{E}}) \right\}.$$

By the similar arguments in the proof of Proposition 4, g^* can be regarded as a Nash equilibrium allocation in the auxiliary economy $\hat{\mathcal{E}}$ with non-consumable resources. g^* is Pareto optimal in $\hat{\mathcal{E}}$ by Propsition 3. This implies that there exists $\lambda \in \mathbb{R}^n_+$ such that

$$g^* \in \arg\max \left\{ \sum_{h \in H} \lambda_h v_h(g) \;\middle|\; g \in \mathcal{A}(\hat{\mathcal{E}}) \right\}.$$

In general, λ is not colinear with $(1, \ldots, 1)$ for a Pareto optimal allocation in $\hat{\mathcal{E}}$. Furthermore any Pareto optimal allocation is attainable as Nash equilibrium with transfer by Corollary 1. We do not guarantee g^* always maximizes $\sum_{h \in H} v_h(g)$ over $\mathcal{A}^{\bar{x}}(\tilde{\mathcal{E}})$. This is why a Nash equilibrium allocation in $\tilde{\mathcal{E}}$ is not the modified version of constrained optimal in general.

8 Concluding Remarks

We study Nash equilibrium allocations of voluntary contribution of non-consumable private resources in an economy with multiple public goods. We give us the several sufficient conditions for Nash equilibrium allocations to be Pareto optimal (the first welfare theorem). We provide us with also the necessary and sufficient condition for Pareto optimal allocations to be Nash equilibrium allocations with transfer (the second welfare theorem). We point out that both the first and the second welfare theorems always hold when all the agents' preferences for public goods bundle are identical, or the number of public goods is two (Proposition 1, 3 and Corollary 1, 2). The economy with identical preferences or with two public goods may be a special case. We cannot assume these settings without loss of generality.

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